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ESTIMATION OF PROPORTIONS IN MIXED PIXELS THROUGH THEIR REGION CHARACTERIZATION

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C. B. Chittineni

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Lockheed Engineering and Management Services Company, Inc.
1830 NASA Road 1, Houston, Texas 77058

Lyndon B. Johnson Space Center
Houston, Texas 77058

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16. Abstract

The estimation of proportions of classes in the mixed pixels of multichannel imagery data is considered in this paper. A significant portion of the imagery data consists of a mixture of the responses of two or more objects whenever the objects being viewed by a multispectral scanner are not large enough relative to the size of a resolution element. A region of mixed pixels can be characterized through the probability density function of proportions of classes in the mixed pixels. Using information from the spectral vectors of a given set of pixels from the mixed pixel region, expressions are developed for obtaining the maximum likelihood estimates of the parameters of probability density functions of proportions. The proportions of classes in the mixed pixels can then be estimated. If the mixed pixels contain objects of two classes, the computation can be considerably reduced by transforming the spectral vectors using a transformation matrix that simultaneously diagonalizes the covariance matrices of the two classes. In addition to the spectral vectors, if the proportions of the classes of a set of mixed pixels from the region are given, then expressions are developed for obtaining the estimates of the parameters of the probability density function of the proportions of mixed pixels. Development of these expressions is based on the criterion of the minimum sum of squares of errors. Furthermore, experimental results from the processing of remotely sensed agricultural multispectral imagery data are presented.

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ESTIMATION OF PROPORTIONS IN MIXED PIXELS THROUGH THEIR REGION CHARACTERIZATION

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This report describes Area Estimation Research activities of the Supporting Research project of the AgRISTARS program.

PREPARED BY

C. B. Chittineni

APPROVED BY

T. C. Minter, Supervisor
Techniques Development Section

J. E. Wainwright, Manager
Development and Evaluation Department

LOCKHEED ENGINEERING AND MANAGEMENT SERVICES COMPANY, INC.

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PREFACE

The techniques which are the subject of this report were developed to support the Agriculture and Resources Inventory Surveys Through Aerospace Remote Sensing program. Under contract NAS 9-15800, Dr. C. B. Chittineni, a principal scientist for Lockheed Engineering and Management Services Company, Inc., performed this research for the Earth Resources Research Division, Space and Life Sciences directorate, National Aeronautics and Space Administration, at the Lyndon B. Johnson Space Center, Houston, Texas. Mrs. Kathy McIntyre provided the programming support for accessing the merged data and ground-truth label files on tape.

CONTENTS

1. INTRODUCTION

Recently, considerable interest has been shown in developing techniques (refs. 1 and 2) for the analysis of multichannel imagery data (such as remotely sensed multispectral scanner data acquired by the Landsat series of satellite) for inventorying natural resources, predicting crop yields, detecting mineral and oil deposits, etc. One of the important objectives in the analysis of remotely sensed imagery data is to estimate the proportion of the crop of interest in the image. Nonsupervised classification or clustering techniques (ref. 2) which partition the image into its inherent modes or clusters have been found to be effective in the classification of imagery data for proportion estimation.

Usually, agricultural imagery data have a field-like structure (ref. 3). The resolution element or pixel of the remote sensing imagery corresponds to approximately 0.44 hectares (1.1 acres) on the ground. A significant portion of the imagery data will contain mixture pixels (i.e., pixels containing objects from more than one class) whenever the objects being viewed by multispectral scanner (MSS) are not large enough relative to the size of a resolution element. The percentage of mixture pixels in the image depends in general on the size of the fields. By analyzing a number of remotely sensed multispectral agricultural images, Nalepka and Hyde (ref. 4) have estimated that, for 20-acre fields, the percentage of mixture pixels in the image is around 40 percent; and, for fields between 60 acres and 100 acres, the percentage of mixture pixels exceeded 20 percent. Hence, to be able to accurately estimate the proportion of the crop of interest in the image, it is necessary to deal with the mixture pixels.

Recently, several researchers (refs. 5 and 6) have attempted to partition or segment a multichannel image into pure pixel (i.e., pixels containing objects of a single class) regions or fields and into mixed pixel or boundary pixel regions. There is considerable interest in developing techniques (refs. 4 and 7) for estimating the proportion of classes in the mixed pixels. In all the proposed methods the proportions of classes in the mixed pixels are estimated

as follows. Assuming the spectral response vector of the mixed pixel as Gaussian, the proportions of classes in the mixed pixel are estimated as those that maximize the likelihood of occurrence of its spectral response vector. One of the reasons these approaches are not successful, in general, is that the individual observation vectors are noisy. In this paper, techniques are developed for estimating the proportions in the mixed pixels by the characterization of region of mixed pixels. The probability density function of the proportion of classes in the mixed pixels is estimated using information from the spectral vectors of a set of mixed pixels from the mixed pixel region. Estimates for the proportion of classes in the mixed pixels are then obtained.

This paper is organized in the following manner. In section 2, relationships are developed between the moments of the spectral vectors of the mixed pixels and the moments of the spectral vectors if the entire pixels contain objects from a single class. In section 3, expressions are developed for the maximum likelihood estimates of the parameters of the probability density function of proportions of classes in the mixed pixels of the region when the mixed pixels contain two classes of objects. Experimental results from the processing of remotely sensed agricultural imagery data are presented in section 4. In section 5, the results of section 3 are generalized when the mixed pixels contain more than two classes of objects. The concluding summary is given in section 6. In the appendix A, using information from the spectral vectors and the proportion of classes of a given set of mixed pixels from the region, expressions for the estimation of the probability density function of the proportion of classes in the mixed pixels of a region are presented. The dependencies between the spectral response vectors of the subpixels of the classes are dealt with in appendix B.

2. A MODEL FOR THE CHARACTERIZATION OF BOUNDARY PIXEL REGIONS

A model for the characterization of regions of boundary pixels is presented in this section. With the present resolution, a pixel or a resolution element of Landsat MSS images represents an acre of ground area. It is observed that more than 30 percent of the pixels of a typical MSS image are boundary or mixed pixels (i.e., pixels containing more than one class of objects).

Let a pixel consist of K small cells of equal size, and let K_i be the number of cells containing the ith class. Let x_{ij} be a random vector representing the spectral response of class i in the jth subcell of these K_i cells. The situation is illustrated in the figure 2-1, where for convenience the subcells of class i are shown as a contiguous block.

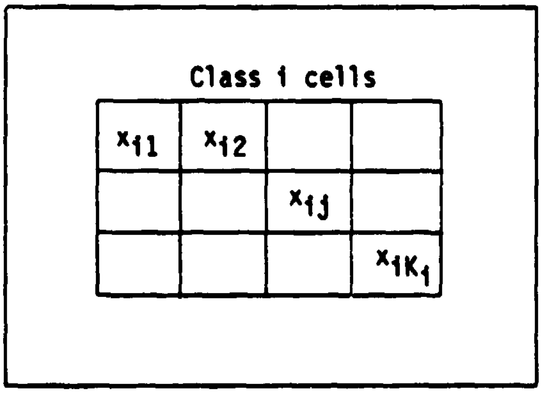

Figure 2-1.- Spectral response vectors associated with the cells of class i in a resolution element.

Let the spectral response vectors x_{ij}, j = 1, 2, ···, K_i, have mean M_i' and covariance matrix Σ_i' for i = 1, 2, ···, R, where R is the number of classes of objects in the resolution element. Let the total response for the resolution element be represented by the random vector X. Assume that X can be written as

$$X = \sum_{i=1}^{R} \sum_{j=1}^{K_i} x_{ij} \tag{2-1}$$

Let K be the total number of subcells of the resolution element, where

$$K = \sum_{i=1}^{R} K_i \tag{2-2}$$

If the entire resolution element were to consist of class i, assuming independence between the spectral response vectors of the subcells,[1] the mean vector M_i and the covariance matrix Σ_i of X can be obtained as follows.

$$\left.\begin{aligned} M_i &= E(X) = KM_i' \\ \text{and} \qquad \Sigma_i &= \mathrm{cov}(X) = K\Sigma_i' \end{aligned}\right\} \tag{2-3}$$

Since there are actually K_i subcells of the ith class, the mean of X is

$$\begin{aligned} E(X) &= \sum_{i=1}^{R} K_i M_i' = \sum_{i=1}^{R} \alpha_i K M_i' \\ &= \sum_{i=1}^{R} \alpha_i M_i = M(\alpha) \end{aligned} \tag{2-4}$$

where

$$\alpha_i = \frac{K_i}{K} \tag{2-5}$$

and is the proportion of class i in the resolution element. The proportions α_i satisfy the following relationships.

$$\left.\begin{aligned} &\alpha_i > 0 \;;\; i = 1, 2, \cdots, R \\ \text{and} \qquad &\sum_{i=1}^{R} \alpha_i = 1 \end{aligned}\right\} \tag{2-6}$$

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[1] The dependencies between the spectral response vectors of the subpixels of the classes are dealt with in appendix B.

If the random vectors associated with the subcells of different classes are also assumed to be independent, the covariance matrix of X can be written as

$$\text{cov}(X) = \sum_{i=1}^{R} K_i \Sigma_i'$$

$$= \sum_{i=1}^{R} \alpha_i \Sigma_i = \Sigma(\alpha) \qquad (2\text{-}7)$$

Let the elements α_i, $i = 1, 2, \cdots, R$, of the vector α satisfy equation (2-6). Let $p(\alpha)$ be the probability density function of α characterizing a region of mixed pixels. Let Ω_α be the region of α in which the constraints of equation (2-6) are satisfied. Let $p_m(X)$ be the probability density function of the spectral response vectors X of the mixed pixels. It can be written as

$$p_m(X) = \int_{\Omega_\alpha} p_m(X,\alpha)d\alpha$$

$$= \int_{\Omega_\alpha} p_m(X|\alpha)p(\alpha)d\alpha \qquad (2\text{-}8)$$

One of the important objectives in the analysis of remotely sensed imagery data is to estimate the proportion of the class of interest in the image. If $p(\alpha)$ is known or estimated, given an observation vector X of a mixed pixel, the Bayes posteriori estimate for the proportion of classes in the mixed pixel can be obtained as follows.

$$\hat{\alpha} = E(\alpha|X)$$

$$= \int_{\Omega_\alpha} \alpha p_m(\alpha|X)d\alpha$$

$$= \frac{\int_{\Omega_\alpha} \alpha p_m(X|\alpha)p(\alpha)d\alpha}{\int_{\Omega_\alpha} p_m(X|\alpha)p(\alpha)d\alpha} \qquad (2\text{-}9)$$

3. ESTIMATION OF $p(\alpha)$ WHEN THE MIXED PIXELS CONTAIN TWO CLASSES OF OBJECTS

The problem of estimation of $p(\alpha)$ to characterize a region of mixed pixels, given the spectral response vectors of a set of mixed pixels from the region, is considered in this section. Very often the proportion of classes in the mixed pixels is unknown. The identification of mixed or border pixels, however, can be obtained by using either the clustering algorithms or the segmentation algorithms. Assuming functional forms for $p(\alpha)$, expressions are developed in the following paragraphs for obtaining the maximum likelihood estimates of the parameters of $p(\alpha)$ using information from the observation vectors of a set of mixed pixels. From the analysis of several ground-truth images, it is observed that suitable functional forms for $p(\alpha)$ are (a) the beta distribution function and (b) the density function representing the portion of a Gaussian curve in the region of interest. These functional forms are described in the following paragraphs. Let α be the proportion of class 1 in the mixed pixel. Then $(1 - \alpha)$ is the proportion of class 2.

a. Beta distribution: Modeling $p(\alpha)$ as a beta distribution in terms of unknown parameters, it can be written as

$$p(\alpha) = \begin{cases} A\alpha^{b}(1-\alpha)^{c} & ; \quad 0 < \alpha < 1 \\ 0 & ; \quad \text{elsewhere} \end{cases} \tag{3-1}$$

where $b > -1$ and $c > -1$ are the parameters to be estimated and the constant A is given by

$$A = \frac{\Gamma(b + c + 2)}{\Gamma(b + 1)\Gamma(c + 1)} \tag{3-2}$$

and $\Gamma(\cdot)$ is a usual gamma function.

b. Gaussian surface: The probability density function $p(\alpha)$ can also be modeled as a portion of Gaussian surface in the allowable region of α. That is, $p(\alpha)$ can be written as

$$p(\alpha) = \begin{cases} \dfrac{f(\alpha)}{\int_0^1 f(\xi)\, d\xi} & ; \quad \text{if } 0 < \alpha < 1 \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{3-3}$$

where $f(\alpha)$ is a Gaussian density function with mean m_f and variance S_f. The parameters m_f and S_f are to be estimated. The probability density function $p(\alpha)$ is illustrated in figure 3-1.

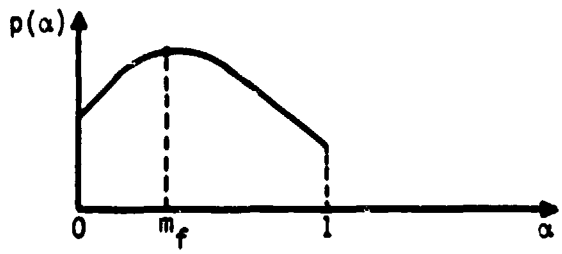

Figure 3-1.- The probability density function $p(\alpha)$, when modeled as the portion of a Gaussian surface.

3.1 MAXIMUM LIKELIHOOD ESTIMATION OF $p(\alpha)$

The estimation of the parameters of $p(\alpha)$ is formulated in this section as that of a maximum likelihood estimation problem. It is assumed that the spectral response vectors X_ℓ, $\ell = 1, 2, \cdots, N$, of a set of mixed pixels are given. The log likelihood of the occurrence of the set of given observation vectors can be written as follows.

$$L = \sum_{i=1}^{N} \log[p_m(X_i)]$$

$$= \sum_{i=1}^{N} \log\left[\int_0^1 p_m(X_i|\alpha)p(\alpha)d\alpha\right] \tag{3-4}$$

Closed form solutions for the parameters of $p(\alpha)$ that maximize L seem to be difficult when the functional form of equation (3-1) or equation (3-3) is used for $p(\alpha)$. In general, the parameters of $p(\alpha)$ that maximize L can be obtained using optimization techniques such as the Davidon-Fletcher-Powell procedure (refs. 8 and 9). However, iterative equations, which are similar to maximum likelihood equations in clustering (refs. 10 and 11), for the estimation of parameters of $p(\alpha)$, can be obtained using the functional form for $p(\alpha)$ given by equation (3-3). The following maximum likelihood equations can easily be derived by differentiating L with respect to the parameters of $p(\alpha)$ and equating the resulting expressions to zero.

$$m_f = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{\int_0^1 \alpha p_m(X_i|\alpha)f(\alpha)d\alpha}{\int_0^1 p_m(X_i|\alpha)f(\alpha)d\alpha}\right] - \left[\frac{\int_0^1 (m_f - \alpha)f(\alpha)d\alpha}{\int_0^1 f(\alpha)d\alpha}\right] \tag{3-5}$$

and

$$S_f = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{\int_0^1 (\alpha - m_f)^2\, p_m(X_i|\alpha)f(\alpha)d\alpha}{\int_0^1 p_m(X_i|\alpha)f(\alpha)d\alpha}\right] + \frac{\int_0^1 \left[S_f - (\alpha - m_f)^2\right]f(\alpha)d\alpha}{\int_0^1 f(\alpha)d\alpha} \tag{3-6}$$

The use of equations (3-5) and (3-6) requires, in general, that the integration be performed numerically. From equations (2-4) and (2-7), for a particular α, the mean and the covariance matrix of the spectral vectors of the mixed pixels are given by the following.

$$M(\alpha) = \alpha M_1 + (1 - \alpha)M_2 \tag{3-7}$$

$$\Sigma(\alpha) = \alpha\Sigma_1 + (1 - \alpha)\Sigma_2 \tag{3-8}$$

The resolution elements that contain a single class are called pure pixels. In the following equations, it is assumed that the spectral vectors of the pure pixels are Gaussian. For a given α, it is also assumed that the spectral vectors of the mixed pixels are Gaussian. In the estimation of m_f and S_f, by iteratively using equations (3-5) and (3-6), the computation can be considerably reduced by transforming the spectral vectors with a transformation matrix that simultaneously diagonalizes the covariance matrices Σ_1 and Σ_2. Let A be the transformation matrix. Then we have [12]

$$\left.\begin{aligned} A\Sigma_1A^T &= I \\ \text{and} \qquad A\Sigma_2A^T &= \Lambda \end{aligned}\right\} \tag{3-9}$$

where $A^T = \Phi\Theta^{-1/2}\psi$. The matrices Θ and Φ are the eigenvalue and eigenvector matrices of Σ_1. The matrices Λ and ψ are the eigenvalue and eigenvector matrices of K, where

$$K = \Theta^{-1/2}\Phi^T\Sigma_2\Phi\Theta^{-1/2} \tag{3-10}$$

Let the spectral vectors X_ℓ be transformed into vectors Y_ℓ, where

$$Y_\ell = AX_\ell \quad ; \quad \ell = 1, 2, \cdots, N \tag{3-11}$$

Let the means M_i of the pattern classes be transformed into μ_i, where

$$\mu_i = AM_i \quad ; \quad i = 1, 2 \tag{3-12}$$

From equations (3-7), (3-8), (3-9), and (3-12), for a given α, the mean and the covariance matrix of the transformed spectral vectors of the mixed pixels are given by the following.

$$\mu(\alpha) = \alpha\mu_1 + (1 - \alpha)\mu_2 \tag{3-13}$$

and

$$S(\alpha) = \alpha I + (1 - \alpha)\Lambda \tag{3-14}$$

The use of $p_m(Y_j|\alpha)$ in equations (3-5) and (3-6) reduces the computation considerably since the determinant and the inverse of matrix $S(\alpha)$ can be computed directly from equation (3-14). An estimate for the proportion of the class of interest (say class 1) in a mixed pixel with the transformed observation vector Y is given by the following.

$$\hat{\alpha} = \frac{\int_0^1 \alpha p_m(Y|\alpha)f(\alpha)d\alpha}{\int_0^1 p_m(Y|\alpha)f(\alpha)d\alpha} \tag{3-15}$$

3.2 MAXIMUM LIKELIHOOD ESTIMATION OF $p(\alpha)$, WITH THE CRITERION OF A LOWER BOUND ON L

It is observed that in equations (3-5) and (3-6) the numerical integration is to be performed at each iteration for the transformed spectral vector of every

given mixed pixel. In the following paragraphs, it is shown that the computation can be considerably simplified by using a lower bound on the likelihood function as a criterion. By noting that the logarithm is a convex upward function, a lower bound on L of equation (3-4) can be obtained as follows.

$$L \geq L_1 \tag{3-16}$$

where

$$L_1 = \int_0^1 A(\alpha)p(\alpha)\, d\alpha \tag{3-17}$$

and

$$A(\alpha) = \sum_{i=1}^{N} \log[p_m(Y_i|\alpha)] \tag{3-18}$$

3.2.1 MAXIMUM LIKELIHOOD EQUATIONS FOR THE ESTIMATION OF PARAMETERS OF $p(\alpha)$

The maximum likelihood equations for the estimation of parameters of $p(\alpha)$ that maximize L_1 of equation (3-17) can easily be shown to be the following.

$$m_f = \frac{\int_0^1 \alpha A(\alpha)f(\alpha)d\alpha}{\int_0^1 A(\alpha)f(\alpha)d\alpha} + \frac{\int_0^1 (m_f - \alpha)f(\alpha)d\alpha}{\int_0^1 f(\alpha)d\alpha} \tag{3-19}$$

and

$$S_f = \frac{\int_0^1 (\alpha - m_f)^2 A(\alpha)f(\alpha)d\alpha}{\int_0^1 A(\alpha)f(\alpha)d\alpha} + \frac{\int_0^1 \left[S_f - (\alpha - m_f)^2\right]f(\alpha)d\alpha}{\int_0^1 f(\alpha)d\alpha} \tag{3-20}$$

It is seen that the use of equations (3-19) and (3-20) requires the integration to be performed numerically, once for every iteration. In the transformed space, an expression for $A(\alpha)$ is given by the following.

$$A(\alpha) = \frac{-nN}{2}\log(2\pi) - \frac{N}{2}\log|S(\alpha)| - \frac{1}{2}tr[S^{-1}(\alpha)SV]$$

$$+ SM^T S^{-1}(\alpha)\mu(\alpha) - \frac{N}{2}\mu^T(\alpha)S^{-1}(\alpha)\mu(\alpha)$$

$$= \frac{-nN}{2}\log(2\pi) - \frac{N}{2}\sum_{i=1}^{n}\log[\alpha + (1-\alpha)\lambda_i] + \sum_{i=1}^{n}\frac{a_i\alpha^2 + b_i\alpha + c_i}{[\alpha + (1-\alpha)\lambda_i]} \tag{3-21}$$

where

$$\left.\begin{aligned} SM &= \left(\sum_{i=1}^{N} Y_i\right) \\ SV &= \left(\sum_{i=1}^{N} Y_i Y_i^T\right) \\ a_i &= \frac{-N}{2}(\mu_{1i} - \mu_{2i})^2 \\ b_i &= (\mu_{1i} - \mu_{2i})[SM(i) - N\mu_{2i}] \\ c_i &= \mu_{2i}\left[SM(i) - \frac{N}{2}\mu_{2i}\right] - \frac{1}{2}\,SV(i,i) \end{aligned}\right\} \quad (3\text{-}22)$$

The diagonal elements of the eigenvalue matrix Λ are λ_i, and the dimensionality of the patterns is n.

3.2.2 CLOSED FORM EXPRESSIONS FOR THE INTEGRALS IN EQUATIONS (3-19) AND (3-20), WHEN THE COVARIANCE MATRICES OF THE CLASSES ARE EQUAL

In the following paragraphs, expressions are derived for the computation of the integrals in equations (3-19) and (3-20) when the covariance matrices of the classes are equal. If the covariance matrices of the classes are equal, then $\lambda_i = 1$ for all i and $A(\alpha)$ in equation (3-21) becomes

$$A(\alpha) = a\alpha^2 + b\alpha + c \quad (3\text{-}23)$$

where

$$\left.\begin{aligned} a &= \sum_{i=1}^{n} a_i \\ b &= \sum_{i=1}^{n} b_i \\ \text{and} \quad c &= \sum_{i=1}^{n} c_i - \frac{nN}{2}\log(2\pi) \end{aligned}\right\} \quad (3\text{-}24)$$

Let

$$\phi(\beta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\beta} \exp\left(-\frac{1}{2}\xi^2\right) d\xi \tag{3-25}$$

The following can now easily be derived.

$$\int_0^1 f(\alpha)d\alpha = \phi\left(\frac{1 - m_f}{\sqrt{S_f}}\right) - \phi\left(\frac{-m_f}{\sqrt{S_f}}\right) \tag{3-26}$$

$$\int_0^1 (\alpha - m_f) f(\alpha)d\alpha = \sqrt{\frac{S_f}{2\pi}} \left\{ \exp\left(\frac{-m_f^2}{2S_f}\right) - \exp\left[-\frac{(1 - m_f)^2}{2S_f}\right] \right\} \tag{3-27}$$

$$\int_0^1 (\alpha - m_f)^2 f(\alpha)d\alpha = \frac{S_f}{\sqrt{2\pi}} \left\{ \frac{-m_f}{\sqrt{S_f}} \exp\left(\frac{-m_f^2}{2S_f}\right) - \frac{(1 - m_f)}{\sqrt{S_f}} \exp\left[-\frac{(1 - m_f)^2}{2S_f}\right] \right\} + \int_0^1 f(\alpha)d\alpha \tag{3-28}$$

$$\int_0^1 (\alpha - m_f)^3 f(\alpha)d\alpha = \frac{S_f^{3/2}}{\sqrt{2\pi}} \left\{ \left(2 + \frac{m_f^2}{S_f}\right) \exp\left(\frac{-m_f^2}{2S_f}\right) - \left[2 + \frac{(1 - m_f)^2}{S_f}\right] \exp\left[-\frac{(1 - m_f)^2}{2S_f}\right] \right\} \tag{3-29}$$

$$\int_0^1 (\alpha - m_f)^4 f(\alpha)d\alpha = \frac{S_f^2}{\sqrt{2\pi}} \left\{ -\left(\frac{3m_f}{\sqrt{S_f}} + \frac{m_f^3}{S_f^{3/2}}\right) \exp\left(\frac{-m_f^2}{2S_f}\right) - \left[\frac{3(1 - m_f)}{\sqrt{S_f}} + \frac{(1 - m_f)^3}{S_f^{3/2}}\right] \right.$$

$$\left. \exp\left[-\frac{(1 - m_f)^2}{2S_f}\right] \right\} + S_f^2 \int_0^1 f(\alpha)d\alpha \tag{3-30}$$

The integrals in equations (3-19) and (3-20) involving the term $A(\alpha)$ can be expressed in terms of the above equations as follows.

$$\int_0^1 A(\alpha)f(\alpha)d\alpha = a \int_0^1 (\alpha - m_f)^2 f(\alpha)d\alpha + 2(am_f + b) \int_0^1 (\alpha - m_f)f(\alpha)d\alpha$$
$$+ \left(am_f^2 + bm_f + c\right) \int_0^1 f(\alpha)d\alpha \tag{3-31}$$

$$\int_0^1 \alpha A(\alpha)f(\alpha)d\alpha = a \int_0^1 (\alpha - m_f)^3 f(\alpha)d\alpha + (3am_f + b) \int_0^1 (\alpha - m_f)^2 f(\alpha)d\alpha$$
$$+ \left(3am_f^2 + 2bm_f + c\right) \int_0^1 (\alpha - m_f)f(\alpha)d\alpha + \left(am_f^3 + bm_f^2 + cm_f\right) \int_0^1 f(\alpha)d\alpha \tag{3-32}$$

$$\int_0^1 (\alpha - m_f)^2 A(\alpha)f(\alpha)d\alpha = a \int_0^1 (\alpha - m_f)^4 f(\alpha)d\alpha + (2am_f + b) \int_0^1 (\alpha - m_f)^3 f(\alpha)d\alpha$$
$$+ \left(am_f^2 + bm_f + c\right) \int_0^1 (\alpha - m_f)^2 f(\alpha)d\alpha \tag{3-33}$$

4. EXPERIMENTAL RESULTS

In this section, some results from the processing of remotely sensed MSS imagery data are presented. Several segments[2] were processed in the following manner. For every segment, several acquisitions are acquired and the images are registered. Each acquisition is a 4-channel image. The 4-channel image values are transformed into greenness and brightness space (ref. 13), thus generating a 2-channel image. Two classes are considered. Class 1 is wheat and class 2 is pasture. The class of interest in the image is wheat. The resolution element or pixel of the image corresponds to approximately an acre on the ground. Each pixel is divided into six subpixels, and the true class labels, or the ground-truth labels for each of the subpixels, are acquired. The pixels containing only wheat, the pixels containing only pasture, and the mixed pixels having wheat and pasture in different proportions are located in the segment. The spectral response vectors of pure pixels are assumed to be Gaussian. For a given α, the spectral response vectors of the mixed pixels are also assumed to be Gaussian. Assuming the functional form of equation (3-3) for $p(\alpha)$, the maximum likelihood estimators for the parameters of $p(\alpha)$ are obtained using equations (3-5) and (3-6). The spectral vectors are transformed using a transformation matrix that simultaneously diagonalizes the covariance matrices of the two classes. Simpson's rule is used for computing the integrals numerically. The proportion of classes of interest (i.e., wheat) in the mixed pixels is estimated using equation (3-15). The number of pixels from each of the classes and the number of mixed pixels are listed in table 1. Also included in table 1 is the average true proportion of wheat in the mixed pixels estimated from the ground-truth labels of the subpixels of the mixed pixels. The estimated proportion of wheat in the mixed pixels using equations (3-5), (3-6), and (3-15), after first iteration and after the convergence, are listed in table 1 for $n = 2$ and 4. For a subset of the segments of table 1, the estimated proportion of wheat in the mixed pixels is listed in table 2 for $n = 6$ and in table 3 for $n = 8$. In general, it is observed that the better

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[2] A segment is a 9- by 11-kilometer (5- by 6-nautical-mile) area for which the MSS image is divided into a 117-row by 196-column rectangular array of pixels.

TABLE 1.- ESTIMATES OF PROPORTION OF WHEAT IN MIXED PIXELS FOR n = 2 AND n = 4

Segment	Location (county/state)	No. of patterns			First iteration		Iterative		Ground-truth proportion
		Wheat	Pasture	Mixture	n = 2	n = 4	n = 2	n = 4	
[a]1005	Cheyenne, Colorado	100	100	350	0.4664	0.4973	0.4662	0.4973	0.5543
[a]1032	Wichita, Kansas	100	100	350	0.5091	0.6257	0.5515	0.6263	0.5057
[a]1033	Clark, Kansas	100	100	343	0.4483	0.4377	0.4188	0.4035	0.5121
[a]1060	Sherman, Texas	100	100	350	0.5044	0.5430	0.5065	0.5523	0.5624
[a]1166	Lyon, Kansas	100	100	350	0.5548	0.5474	0.6187	0.6441	0.5100
[a]1231	Jackson, Oklahoma	100	100	350	0.4852	0.4859	0.4607	0.4716	0.5657
[a]1367	Major, Oklahoma	100	100	350	0.4975	0.4967	0.3079	0.4967	0.5524
[b]1512	Clay, Minnesota	100	100	83	0.6370	0.5329	0.6279	0.5541	0.5703
[b]1520	Big Stone, Minnesota	100	100	212	0.4921	0.5340	0.4861	0.5958	0.5464
[b]1544	Sheridan, Montana	100	100	274	0.5389	0.5044	0.6496	0.5135	0.5024
Bias					0.248E-1	0.1767E-1	0.2918E-1	0.265E-2	
MSE					0.354E-2	0.35987E-2	0.1336E-1	0.6245E-2	

[a]Winter wheat segments.
[b]Spring wheat segments.

TABLE 2.- ESTIMATES OF PROPORTION OF WHEAT IN MIXED PIXELS FOR $n = 6$

Segment	Location (county/state)	No. of patterns			First iteration	Iterative	Ground-truth proportion
		Wheat	Pasture	Mixture			
1005	Cheyenne, Colorado	100	100	350	0.4504	0.4402	0.5543
1032	Wichita, Kansas	100	100	350	0.6362	0.7784	0.5057
1166	Lyon, Kansas	100	100	350	0.5480	0.6240	0.5100
1231	Jackson, Oklahoma	100	100	350	0.4027	0.3143	0.5657
1367	Major, Oklahoma	100	100	350	0.5080	0.5189	0.5524
1520	Big Stone, Minnesota	100	100	212	0.5368	0.5970	0.5464
Bias					0.254E-1	-0.6383E-2	
MSE					0.96503E-2	0.2787E-1	

TABLE 3.- ESTIMATES OF PROPORTION OF WHEAT IN MIXED PIXELS FOR n = 8

Segment	Location (county/state)	No. of patterns			First iteration	Iterative	Ground-truth proportion
		Wheat	Pasture	Mixture			
1005	Cheyenne, Colorado	100	100	350	0.4385	0.3807	0.5543
1032	Wichita, Kansas	100	100	350	0.6433	0.7522	0.5057
1166	Lyon, Kansas	100	100	350	0.4969	0.4974	0.5100

proportion estimates are obtained for $n = 4$. It is thought that the degradation in the estimates with the increase in the number of acquisitions is due to the registration errors.

5. ESTIMATION OF $p(\alpha)$ WHEN THE MIXED PIXELS CONTAIN MORE THAN TWO CLASSES OF OBJECTS

The problem of estimation of $p(\alpha)$ when the mixed pixels contain more than two classes of objects is considered in this section. The functional forms that can be used for $p(\alpha)$ are the multivariate generalization of the ones presented in section 3. These are described in the following paragraphs.

a. The Dirichlet Distribution: If $p(\alpha)$ can be represented as a Dirichlet distribution function, it can be written as

$$p(\alpha) = K \prod_{i=1}^{R} \alpha_i^{a_i}$$

$$= K\left(1 - \sum_{j=1}^{R-1} \alpha_j\right) \left[\prod_{i=1}^{R-1} \alpha_i^{a_i}\right] \tag{5-1}$$

where $\sum_{i=1}^{R} \alpha_i = 1$, $\alpha_i \geq 0$ for $i = 1, 2, \cdots, R$, and

$$K = \frac{\Gamma\left[\sum_{i=1}^{R} (a_i + 1)\right]}{\prod_{j=1}^{R} [\Gamma(a_j + 1)]} \tag{5-2}$$

The set of parameters $\{a_i\}$ are such that $a_i > -1$ for $i = 1, 2, \cdots, R$, and are to be estimated.

b. The multivariate Gaussian surface: By modeling $p(\alpha)$ with the surface of a multivariate normal distribution in the region Ω_α, $p(\alpha)$ can be written as

$$p(\alpha) = \begin{cases} \dfrac{f(\alpha)}{\int_{\Omega_\alpha} f(\alpha)d\alpha} & ; \text{ if } \alpha \in \Omega_\alpha \\ 0 & ; \text{ otherwise} \end{cases} \tag{5-3}$$

where $f(\alpha)$ is a Gaussian density function with the mean vector M_f and the covariance matrix Σ_f. The parameters M_f and Σ_f are to be estimated.

5.1 MAXIMUM LIKELIHOOD ESTIMATION OF $p(\alpha)$

Given the spectral response vectors X_ℓ, $\ell = 1, 2, \cdots, N$, of a set of mixed pixels, the log likelihood of the occurrence of the given set of observation vectors can be written as follows.

$$L = \log\left[\prod_{i=1}^{N} p_m(X_i)\right]$$

$$= \sum_{i=1}^{N} \log\left[\int_{\Omega_\alpha} p_m(X_i|\alpha)p(\alpha)d\alpha\right] \tag{5-4}$$

In general, using the functional forms for $p(\alpha)$ that are given either in equation (5-1) or in equation (5-3), the parameters of $p(\alpha)$ that maximize L can be obtained using optimization techniques such as Davidon-Fletcher-Powell (refs. 8 and 9). If the functional form given by equation (5-3) is used for $p(\alpha)$, the following maximum likelihood equations for the estimation of parameters of $p(\alpha)$ that maximize L can easily be derived.

$$M_f = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{\int_{\Omega_\alpha} \alpha p_m(X_i|\alpha)f(\alpha)d\alpha}{\int_{\Omega_\alpha} p_m(X_i|\alpha)f(\alpha)d\alpha}\right] + \left[\frac{\int_{\Omega_\alpha}(M_f - \alpha)f(\alpha)d\alpha}{\int_{\Omega_\alpha} f(\alpha)d\alpha}\right] \tag{5-5}$$

and

$$\Sigma_f = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{\int_{\Omega_\alpha}(\alpha - M_f)(\alpha - M_f)^T p_m(X_i|\alpha)f(\alpha)d\alpha}{\int_{\Omega_\alpha} p_m(X_i|\alpha)f(\alpha)d\alpha}\right] + \frac{\int_{\Omega_\alpha}\left[\Sigma_f - (\alpha - M_f)(\alpha - M_f)^T\right]f(\alpha)d\alpha}{\int_{\Omega_\alpha} f(\alpha)d\alpha} \tag{5-6}$$

It is noted that in equations (5-5) and (5-6) the integrals need to be computed for every spectral vector at each iteration.

5.2 MAXIMUM LIKELIHOOD ESTIMATION OF $p(\alpha)$ WITH THE CRITERION OF A LOWER BOUND ON L

Since the logarithm is a convex upward function, a lower bound on L of equation (5-4) can be obtained as

$$L > L_1 \tag{5-7}$$

where

$$L_1 = \int_{\Omega_\alpha} A(\alpha)p(\alpha)d\alpha \tag{5-8}$$

and

$$A(\alpha) = \sum_{i=1}^{N} \log[p_m(X_i|\alpha)] \tag{5-9}$$

If the functional form given by equation (5-3) is used for $p(\alpha)$, the following maximum likelihood equations for the estimation of parameters of $p(\alpha)$ that maximize L_1 can easily be derived.

$$M_f = \frac{\int_{\Omega_\alpha} \alpha A(\alpha)f(\alpha)d\alpha}{\int_{\Omega_\alpha} A(\alpha)f(\alpha)d\alpha} + \frac{\int_{\Omega_\alpha} (M_f - \alpha)f(\alpha)d\alpha}{\int_{\Omega_\alpha} f(\alpha)d\alpha} \tag{5-10}$$

and

$$\Sigma_f = \frac{\int_{\Omega_f} (\alpha - M_f)(\alpha - M_f)^T A(\alpha)f(\alpha)d\alpha}{\int_{\Omega_\alpha} A(\alpha)f(\alpha)d\alpha} + \frac{\int_{\Omega_\alpha} \left[\Sigma_f - (\alpha - M_f)(\alpha - M_f)^T\right]f(\alpha)d\alpha}{\int_{\Omega_\alpha} f(\alpha)d\alpha} \tag{5-11}$$

It is observed that the use of equations (5-10) and (5-11) requires the integrals to be computed once for every iteration.

6. CONCLUDING SUMMARY

One of the important objectives in the processing of remotely sensed imagery data is the estimation of the proportion of the class of interest in the image. Agricultural imagery data usually have a field-like structure. From the analysis of several images, it is observed that depending on the field size, a significant portion of the image contains mixed pixels (i.e., pixels containing more than one class of objects). Techniques are currently being developed for partitioning an image into regions of pure pixels and mixed pixels.

This paper addresses the problem of estimating the proportions of classes in the mixed pixels. Relationships are developed between the moments of spectral vectors of mixture pixels and the moments of spectral vectors of pure pixels of different classes of objects as a function of the proportion of classes in the mixed pixel. The probability density function $p(\alpha)$ of the proportion of classes in the mixed pixels can be modeled either as a Dirichlet distribution or as a normalized Gaussian surface in the region of interest in terms of unknown parameters. Given the spectral vector of a mixed pixel, the proportion of classes in the pixel can then be estimated using $p(\alpha)$.

By modeling $p(\alpha)$ as a normalized Gaussian surface, expressions are developed for obtaining the maximum likelihood estimates of its parameters using information from the spectral vectors of a set of mixed pixels from the mixed pixel region. These involve evaluation of integrals numerically. If the mixed pixels contain two classes of objects, the computation can be considerably reduced by transforming the spectral vectors using a transformation matrix that simultaneously diagonalizes the covariance matrices of the classes. Closed form expressions are developed for the computation of the integrals when the covariance matrices of the classes are equal.

Experimental results are presented from the processing of remotely sensed agricultural imagery data. Two classes are considered. Class 1 is wheat and class 2 is pasture. The mixture pixels contain wheat and pasture in different proportions. The proportion of wheat in the mixture pixels is estimated using

equations (3-5), (3-6), and (3-15) with the dimensionality of the spectral vector n where $n = 2, 4, 6$, and 8. The proportion of wheat in the mixed pixels is also estimated using the true class labels of the six subpixels of a pixel from the ground truth. Better proportion estimates are obtained for the value of $n = 4$ or with the data from two acquisitions.

Expressions are developed for the estimation of parameters of $p(\alpha)$ based on the criterion of the minimum sum of the squares of errors, using information from the spectral vectors and proportions of classes of a set of mixed pixels from the mixed pixel region. Furthermore, the effect of the dependencies between the spectral vectors of the subpixels of the classes on the moments of the spectral vectors of the mixed pixels is considered in appendix B.

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APPENDIX A

ESTIMATION OF $p(\alpha)$ WITH THE CRITERION OF THE MINIMUM SUM OF THE SQUARES OF ERRORS

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ESTIMATION OF p(α) WITH THE CRITERION OF THE MINIMUM SUM OF THE SQUARES OF ERRORS

The problem of characterization of a region of mixed pixels through the estimation of p(α) using information from the spectral vectors of a given set of pixels from the region was treated in sections 3 and 5. If the proportion of classes in the mixed pixels of the given set are also known, the problem of estimation of p(α) using all the available information (in addition to the spectral vectors) is considered in this appendix. The estimates of the parameters of p(α) are obtained using the minimum sum of the squares of errors as a criterion.

Let X_ℓ, $\ell = 1, 2, \cdots, N$, be the n-dimensional spectral response vectors of the given set of mixed pixels. Let α_i, $i = 1, 2, \cdots, N$, be the R-dimensional vectors of proportions of classes in the mixed pixels. Given the spectral vector X_i of a mixed pixel and p(α), an estimate for the proportions of classes in the pixel is given by

$$\hat{\alpha}_i = \frac{\int_{\Omega_\alpha} \alpha p(X_i|\alpha)p(\alpha)d\alpha}{\int_{\Omega_\alpha} p(X_i|\alpha)p(\alpha)d\alpha} \qquad \text{(A-1)}$$

If the functional form of equation (5-3) is used for the probability density function p(α), equation (A-1) can be written as

$$\hat{\alpha}_i = \frac{\int_{\Omega_\alpha} \alpha p(X_i|\alpha)f(\alpha)d\alpha}{\int_{\Omega_\alpha} p(X_i|\alpha)f(\alpha)d\alpha} \qquad \text{(A-2)}$$

where f(α) is a Gaussian density function with the mean vector M_f and the covariance matrix Σ_f. The criterion of the minimum sum of the squares of errors can be used for obtaining the parameters M_f and Σ_f of p(α). The sum of the squares of errors, ε, of the proportion estimates can be written as follows.

$$\epsilon = \sum_{i=1}^{N} \left(\alpha_i - \hat{\alpha}_i\right)^T \left(\alpha_i - \hat{\alpha}_i\right) \tag{A-3}$$

If θ is a parameter of $p(\alpha)$, differentiating ϵ with respect to θ results in

$$\frac{\partial \epsilon}{\partial \theta} = \sum_{i=1}^{N} \left(\hat{\alpha}_i - \alpha_i\right)^T \frac{\partial \hat{\alpha}_i}{\partial \theta} \tag{A-4}$$

From equation (A-2), we get

$$\frac{\partial \hat{\alpha}_i}{\partial \theta} = \frac{\int_{\Omega_\alpha} \alpha p(X_i|\alpha) \frac{\partial}{\partial \theta} [f(\alpha)]d\alpha - \hat{\alpha}_i \int_{\Omega_\alpha} p(X_i|\alpha) \frac{\partial}{\partial \theta} [f(\alpha)]d\alpha}{\left[\int_{\Omega_\alpha} p(X_i|\alpha)f(\alpha)d\alpha\right]} \tag{A-5}$$

Differentiating $f(\alpha)$ with respect to its mean vector M_f yields

$$\frac{\partial f(\alpha)}{\partial M_f} = \Sigma_f^{-1}(\alpha - M_f)f(\alpha) \tag{A-6}$$

Let ν_{ij} be the elements of the matrix Σ_f^{-1}. Differentiating $f(\alpha)$ with respect to ν_{ij} results in the following.

$$\left.\begin{aligned} \frac{\partial f(\alpha)}{\partial \nu_{ii}} &= \left[\sigma_{ii} - (\alpha_i - M_{fi})^2\right] \frac{f(\alpha)}{2} \\ \text{and} \qquad \frac{\partial f(\alpha)}{\partial \nu_{ij}} &= [\sigma_{ij} - (\alpha_i - M_{fi})(\alpha_j - M_{fj})]f(\alpha) \end{aligned}\right\} \tag{A-7}$$

Where σ_{ij} are the elements of the matrix Σ_f and M_{fi} is the ith element of the vector M_f. Substitution of equations (A-5), (A-6), and (A-7) in equation (A-4) yields iterative equations (A-8) and (A-9), which are similar to maximum likelihood equations (refs. 10 and 11), for the estimation of parameters M_f and Σ_f of $p(\alpha)$.

When there are only two classes in the mixed pixel, as shown in section 3, the computation can be greatly reduced by transforming the spectral vectors using a transformation matrix that simultaneously diagonalizes the covariance matrices of the classes.

$$M_f = \frac{1}{\left[\sum_{i=1}^{N} \left(\hat{\alpha}_i - \alpha_i\right)^T \alpha_i\right]} \cdot \left\{ \sum_{i=1}^{N} \frac{\int_{\Omega_\alpha} \left[\hat{\alpha}_i - \alpha_i\right]^T \alpha \, p(X_i | \alpha) f(\alpha) d\alpha}{\int_{\Omega_\alpha} p(X_i | \alpha) f(\alpha) d\alpha} + \sum_{i=1}^{N} \left[\left(\hat{\alpha}_i - \alpha_i\right)^T \hat{\alpha}_i\right] \cdot \frac{\int_{\Omega_\alpha} (M_f - \alpha) p(X_i | \alpha) f(\alpha) d\alpha}{\int_{\Omega_\alpha} p(X_i | \alpha) f(\alpha) d\alpha} \right\} \tag{A-8}$$

$$\Sigma_f = \frac{1}{\left[\sum_{i=1}^{N} \left(\hat{\alpha}_i - \alpha_i\right)^T \hat{\alpha}_i\right]} \cdot \left[\sum_{i=1}^{N} \frac{\int_{\Omega_\alpha} \left[(\alpha - M_f)(\alpha - M_f)^T\right]\left[(\hat{\alpha}_i - \alpha_i)^T \alpha\right] p(X_i | \alpha) f(\alpha) d\alpha}{\int_{\Omega_\alpha} p(X_i | \alpha) f(\alpha) d\alpha} + \sum_{i=1}^{N} \left[\left(\hat{\alpha}_i - \alpha_i\right)^T \hat{\alpha}_i\right] \cdot \frac{\int_{\Omega_\alpha} \left[\Sigma_f - (\alpha - M_f)(\alpha - M_f)^T\right] p(X_i | \alpha) f(\alpha) d\alpha}{\int_{\Omega_\alpha} p(X_i | \alpha) f(\alpha) d\alpha} \right] \tag{A-9}$$

APPENDIX B

EFFECT OF CORRELATIONS BETWEEN THE SPECTRAL VECTORS OF SUBPIXELS ON THE MOMENTS OF SPECTRAL VECTORS OF MIXED PIXELS

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In section 2, it is assumed that the spectral vectors of the subpixels are independent. The purpose of this appendix is to take into account the correlations between the spectral vectors of subpixels in developing expressions for the moments of the spectral vectors of the mixture pixels. If the entire resolution element were to consist of class i, the spectral vector X of the resolution element can be written in terms of the spectral vectors of the subpixels as

$$X = \sum_{j=1}^{K} x_{ij} \tag{B-1}$$

The mean vector M_i and the covariance matrix Σ_i of X can be obtained as follows.

$$M_i = E(X) = KM_i' \tag{B-2}$$

$$\begin{aligned}
\Sigma_i &= \mathrm{cov}(X) \\
&= E\left\{\left[\sum_{j=1}^{K} (x_{ij} - M_i')\right]\left[\sum_{j=1}^{K} (x_{ij} - M_i')^T\right]\right\} \\
&= E\left[\sum_{j=1}^{K} (x_{ij} - M_i')(x_{ij} - M_i')^T + \sum_{j=1}^{K} \sum_{\substack{k=1 \\ k \neq j}}^{K} (x_{ij} - M_i')(x_{ik} - M_i')^T\right] \\
&= K\Sigma_i' + \sum_{j=1}^{K} \sum_{\substack{k=1 \\ k \neq j}}^{K} E\left[(x_{ij} - M_i')(x_{ik} - M_i')^T\right]
\end{aligned} \tag{B-3}$$

If the spectral vectors of the subpixels are independent, the second term on the right-hand side of equation (B-3) becomes zero. Let

$$Z_{isr} = \left(x_{is}^T, x_{ir}^T\right)^T \tag{B-4}$$

Let Σ'_{iz} be the covariance matrix of the random vector Z_{isr}, which can be written as

$$\Sigma'_{iz} = \begin{bmatrix} \Sigma'_i & \Sigma'_{isr} \\ \Sigma'^{T}_{isr} & \Sigma'_i \end{bmatrix} \tag{B-5}$$

Let

$$\Sigma'^{-1}_{iz} = \begin{bmatrix} Q_i & Q_{isr} \\ Q^T_{isr} & Q_i \end{bmatrix} \tag{B-6}$$

If the random vectors x_{ir} and x_{is} are Gaussian with mean M'_i and covariance matrix Σ'_i, the conditional probability density $p(x_{is}|x_{ir})$ is normal with mean vector $M'_i - Q^{-1}_{is}Q_{isr}(x_{ir} - M'_i)$ and covariance matrix Q^{-1}_i. Now consider

$$E\left[(x_{ir} - M'_i)(x_{is} - M'_i)^T\right] = \int(x_{ir} - M'_i)\left[\int(x_{is} - M'_i)^T p(x_{is}|x_{ir})dx_{is}\right]p(x_{ir})dx_{ir} \tag{B-7}$$

Using equation (B-7) in equation (B-3) yields

$$\Sigma_i = K\Sigma'_i - \Sigma'_i \sum_{r=1}^{K} \sum_{\substack{s=1 \\ s\neq r}}^{K} Q^T_{isr}Q^{-1}_{is} \tag{B-8}$$

It is assumed that the covariance matrix Σ'_{isr} of radiance vectors x_{is} and x_{ir} can be written as

$$\Sigma'_{isr} = a_{sr}\Sigma'_i \tag{B-9}$$

where a_{sr} is a constant which may depend on the spatial distance between the r^{th} and the s^{th} subpixels. Using equations (B-5), (B-6), and (B-9) in equation (B-8) yields

$$\Sigma_i = K\Sigma'_i + \left(\sum_{r=1}^{K} \sum_{\substack{s=1 \\ s\neq r}}^{K} a_{sr}\right)\Sigma'_i \tag{B-10}$$

The quantity within parentheses in equation (B-10), in general, depends on the number of subpixels and the spatial arrangement of subpixels or on the shape of the region of a class in a resolution element. Let δ be a quantity representative of the shape of a region of a class in a resolution element; then, equation (B-10) can be written as

$$\begin{aligned} \Sigma_i &= K\Sigma_i' + \delta K\Sigma_i' \\ &= \nu K\Sigma_i' \end{aligned} \tag{B-11}$$

where
$$\nu = (1 + \delta) \tag{B-12}$$

If there are R-classes and K_i subcells of each class in a resolution element, the spectral vector of the resolution element can be written as

$$X = \sum_{i=1}^{R} \sum_{j=1}^{K_i} x_{ij} \tag{B-13}$$

The mean vector M of X can be obtained as follows. Consider

$$\begin{aligned} M = E(X) &= \sum_{i=1}^{R} \sum_{j=1}^{K_i} E(x_{ij}) \\ &= \sum_{i=1}^{R} K_i M_i' \\ &= \sum_{i=1}^{R} \alpha_i M_i \end{aligned} \tag{B-14}$$

Assuming the spectral response vectors of subpixels of different classes are independent, the covariance matrix of X can be obtained as follows.

$$\Sigma = \mathrm{cov}(X) = E\left\{\left[\sum_{i=1}^{R}\sum_{j=1}^{K_i}(x_{ij} - M_i')\right]\left[\sum_{i=1}^{R}\sum_{j=1}^{K_i}(x_{ij} - M_i')^T\right]\right\}$$

$$= \sum_{i=1}^{R}\left\{\sum_{j=1}^{K_i}\sum_{\ell=1}^{K_i} E\left[(x_{ij} - M_i')(x_{i\ell} - M_i')^T\right]\right\}$$

$$= \sum_{i=1}^{R}(1 + \delta_i)K_i\Sigma_i'$$

$$= \sum_{i=1}^{R}\frac{\nu_i}{\nu}\frac{K_i}{K}\Sigma_i$$

$$= \sum_{i=1}^{R}\beta_i\alpha_i\Sigma_i \tag{B-15}$$

where

$$\left.\begin{aligned}\beta_i &= \frac{\nu_i}{\nu}\\ \nu_i &= (1 + \delta_i)\end{aligned}\right\} \tag{B-16}$$

and δ_i is a quantity representative of the shape of the region of i^{th} class in a resolution element. A comparison of equations (2-7) and (3-15) shows that the effect of correlations between the subpixels of classes is to introduce the constants β_i.

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